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A simple stochastic cluster dynamics: rigorous results

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Abstract. Motivated by the Swendsen–Wang algorithm for the Ising model with no external magnetic field we consider a particle system on a discrete lattice evolving according to the following parallel algorithm: at each time step connected clusters of particles are removed with probability $\frac{1}{2}$ and at each empty site a particle is created with probability p. Due to the presence of arbitrarily large clusters the interaction can have an arbitrary range. In the one-dimensional case we solve the model completely by means of a novel path expansion in spacetime. In the two-dimensional case we show that, due to strong time correlation, the probability of having a large cluster of particles does not rapidly decay exponentially in the cardinality of the cluster even for p very small. We then prove ergodicity for $1-p$ close to zero.

1. Introduction

In the last few years, stimulated by the beautiful work by Swendsen and Wang [1], there has been an increasing interest in computational physics in the so-called stochastic cluster algorithms for the Monte Carlo simulation of models of statistical physics such as the Ising and Potts models, and frustrated systems (see [2–5]).

In these algorithms, contrary to the more traditional single-site algorithm like Glauber dynamics, at each time step one is allowed to change a whole cluster of dynamic variables simultaneously (e.g. the spins in the Ising model). The result is, in general, a better numerical efficiency for the algorithm and a faster convergence of the Monte Carlo simulation.

From a rigorous point of view stochastic cluster algorithms have received little attention and only recently have some partial results for the Swendsen–Wang (SW) dynamics been obtained (see [6–8]). In particular in [7] and [8] in collaboration with E Olivieri we successfully analysed the low temperature behaviour of the SW algorithm for the d-dimensional Ising model in the presence of a small magnetic field. In SW dynamics the effect of an external field h in updating a large cluster of spins produces an almost sure move in the sense that putting the new values of the spins in a cluster C antiparallel to the field has an exponentially small probability in the cardinality of the cluster. This feature allowed us to control the flow of information through the system by means of simple expansions, that is the influence among distant regions and as a consequence the rate of approach to the equilibrium. However the very interesting case of the zero external field remained open. In this case each cluster becomes either plus or minus with probability $\frac{1}{2}$ irrespective of its size and therefore a rigorous analysis of the dynamics becomes much harder.

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Motivated by this problem we consider here as a laboratory a simple stochastic cluster dynamics which shares with the SW algorithm without an external field the property that the probability that the updating of clusters of dynamical variables (particles in our case) occurs is independent of the geometry of the cluster.

The setting is as follows: at each point x in the box $\Lambda = [-L, L]^d \cap \mathbf{Z}^d$ we associate an occupation variable $\sigma(x)$ with values 0 or 1; given a configuration σ_t at time t in order to define the new configuration σ_{t+1} at time $t+1$ we first consider all connected clusters of particles (sites at which the configuration σ_t is equal to 1) and we remove each cluster independently with probability $\frac{1}{2}$; as a second step we create particles in each empty site independently with probability p.

This dynamics is similar to a model considered by Swindle and Grannan in [9] although in their model clusters disappear with a rate proportional to their size. We will be primarily interested in the long time behaviour of this stochastic cluster dynamics and, in particular, in such questions as ergodicity, approach to equilibrium and the mixing properties of the invariant measure. It turns out that, in order to carry out this analysis, it is crucial to control the range of the interaction namely the typical size of the clusters. Thus we will concentrate on estimates of the probability that the origin belongs to a big cluster consisting of n particles. In the one-dimensional case this probability is bounded by a negative exponential in n for any $p \in (0,1)$. As a consequence we can then prove for all p the exponential convergence as t tends to infinity of the distribution of the process at time t to the unique invariant measure together with the exponential decay of correlations of the invariant measure. The proof, based on a novel expansion in spacetime, is quite simple for small values of p but is non-trivial for p close to 1.

In dimensions greater than one the situation changes radically. For any $p \in (0,1)$ we prove that this probability cannot be bounded from above by a negative exponential *in the number of sites of the cluster. More precisely, if* $q_n(t)$ *denotes the probability* that the cube Q_n of side n centred at the origin is filled with particles at time t, then we prove that, for suitable constants c_1, c_2, α, β if $t \geq c_1 N$, we have:

$$q_n(t) \geq \exp(-c_2 n) \qquad \forall p > 0 \tag{1.1a}$$

$$\frac{1}{n^\beta} \geq q_n(t) \geq \frac{1}{n^\alpha} \qquad \text{if } 1-p \ll 1. \tag{1.1b}$$

It follows from this that if $P_V(t)$ denotes the probability that the origin belongs to a cluster of volume V, then $P_V(t) \geq \exp(-V^{1/d})$, i.e. a *subexponential* decay in the size of the cluster. It is important to understand intuitively the reason behind such a striking difference between $d = 1$ and $d \geq 2$ and particularly in the low activity regime $p \ll 1$ which for other systems does not discriminate between different dimensions.

For simplicity we illustrate the argument for the two-dimensional case. Among many others there are two basic possible mechanisms which fill the square Q_n with particles at time t. The simplest one is to create a particle at each site of Q_n at time t. This has probability p^{n^2}. A more sophisticated way is first to create a particle at each site of the boundary ∂Q_n at a previous time t' with $t - t' \geq n/p$ and then to impose the condition that the cluster of particles containing the boundary of Q_n never disappears for any time s between t' and t. It is easy to see that under this last condition the cluster containing the boundary of Q_n with a probability close to one grows under the random dynamics and that it will cover the whole interior of Q_n before time t. Therefore the probability of this second strategy is only of order $(\frac{1}{2})^{(t-t')}p^{4n}$, namely a negative exponential in n and not in n^2, and the system will prefer this second choice over the first one. In one dimension both strategies have probabilities of the same order of magnitude and the phenomenon does not take place. In the case of p close to one the power law bounds (1.1b) can be derived by a similar argument if one observes that, using results from percolation, at each time t and with a probability close to one we create an infinite cluster of particles with holes whose size grows only logarithmically with their distance from the origin. This observation can also be used to prove, in the same range of values of p, the ergodicity and exponential approach to equilibrium. The same problem for p small is not covered by the methods of the present paper. After this paper was completed one of us (FM) envisaged a strategy with which to treat the SW dynamics at low temperature without a magnetic field [10]. Such a strategy also applies to the present model and proves fast convergence to equilibrium and the uniqueness of the invariant measure even for p small. The similarity of the dynamics discussed here to the SW dynamics is, however, less close than it appears at first glance. The SW dynamics is highly non-local, but its invariant measure is the standard nearest-neighbour Ising model. By contrast, the invariant measure of the present model is (at least in dimension $d \geq 2$) *not the Gibbs measure for any absolutely summable interaction!* This follows from the fact that the probability (in the invariant measure) of having a square of side N completely filled with particles decays *subexponentially* in the volume $V = N^d$, whereas in any Gibbs state measure it would have to decay exponentially in V. Similar results are found in [11] for the voter model and in [12] and [13] for the sign of a massless Gaussian field. Our results support the contention of Lebowitz and Schonmann [11] that invariant measures in non-equilibrium statistical mechanics should be expected generically to be *non-Gibbsian.*

2. The model

In this section we define more precisely the model that will be investigated in the present paper and we will prove some simple general results about it that will be quite useful later on.

Let $\Lambda = [-L, L]^d \cap \mathbf{Z}^d$ and let C_Λ be the collection of all possible connected subsets I of Λ. Here I is connected iff for any two sites x and y there exists a path of nearest-neighbours sites in I going from x to y.

Let also $\{\nu(x,s)\}_{x\in\Lambda, s\in\mathbf{N}}$ and $\{\xi(I,s)\}_{I\in C_\Lambda, s\in\mathbf{N}}$ be independent identically distributed (i.i.d.) random variables with values in $\{0,1\}$ with:

$$P(\nu(x,s)=1) = p \qquad P(\xi(I,s)=1) = \tfrac{1}{2}.$$

For brevity a realization of the $\nu(x,s)$ and $\xi(I,s)$ variables will be denoted by ν and ξ respectively. On each site x of Λ we will associate an occupation variable $\sigma(x)$ taking values in $\{0,1\}$; for brevity the collection of the variables $(\sigma(x))_{x\in\Lambda}$ will be denoted by σ. Thus σ is an element of the configuration space $S = \{0,1\}^\Lambda$. Using the random variable ν, ξ we now construct on S a random dynamics starting at the configuration σ at time $t=0$ as follows:

(i) Given $\sigma_t^\Lambda \in S$ we set for any $x \in \Lambda$:

$$\sigma^\Lambda_{t+\frac{1}{2}}(x) = 1 \qquad \text{iff } \sigma_t^\Lambda(x) = 1 \text{ and } \xi(I_x, t) = 1$$

where I_x is the maximal element of C_Λ containing x such that $\sigma_t^\Lambda(y) = 1\ \forall y \in I$.

(ii) For any $x \in \Lambda$:

$$\sigma_{t+1}^\Lambda(x) = 0 \qquad \text{iff } \sigma_{t+\frac{1}{2}}^\Lambda(x) = 0 \text{ and } \nu(t+1, x) = 0.$$

For brevity we will refer to (i) as the annihilation of particles and to (ii) as the creation of particles. Note that both processes occur simultaneously (i.e. the updating is parallel) and that the non-trivial interaction of the model is completely contained in the annihilation process.

We will refer to these rules as the 'basic dynamics in Λ'. The associated Markov process will always be denoted by σ_t omitting the suffix Λ for brevity whenever it does not produce confusion.

It is very easy to check that in any finite volume Λ there exists a unique invariant measure that will be denoted by μ_Λ.

Later on in this article, when discussing the process's approach to equilibrium we will need to compare the dynamics of a given site x produced by two different boxes Λ and Λ' with $\Lambda' \subset \Lambda$ both containing x. This will be done by establishing a coupling between the two dynamics according to the following rules:

(a) The variables $\nu(x, s)_{x \in \Lambda'}$ are exactly the same variables as those chosen for the dynamics in Λ i.e. if a particle is created inside Λ' for the dynamics in Λ then it is also created for the dynamics in Λ' and *vice versa.*

(b) The value of $\xi(I, s)$ is the same for both dynamics if $I \subset \Lambda'$.

In some sense this coupling is the most natural way to restrict the dynamics in Λ to Λ'.

In one dimension, however, there is a more efficient way to realize this coupling in such a way that the value of the process at a given site x inside Λ' will always be equal for the two dynamics. Rule (a) remains unchanged while (b) becomes:

(c) $\xi(I, s) = \xi(I \cap \Lambda', s)$.

In other words a cluster I will disappear or stay according to whether its restriction to Λ' disappears or stays. Clearly such a rule is consistent only in one dimension since in higher dimension $I \cap \Lambda'$ may consist of more than one cluster. It is important to realize that this construction of the dynamics in Λ produces a random flow on S that it is different from the 'basic dynamics in Λ' without, however, affecting the probability of any event since the probability of annihilating a cluster does not depend on its size or shape.

In one dimension we will always make this last choice.

A nice consequence of this coupling is that in one dimension there is always a unique invariant measure μ for the infinite volume process.

Theorem 2.1.

(i) Let Λ be the finite interval $[-L, L] \cap \mathbf{Z}$, let μ_Λ be the associated invariant measure, and let Ω_A be a cylindrical event in S depending only on the value of the configuration σ in $A \subset \Lambda$. Then:

$$\mu_\Lambda(\Omega_A) = \mu_A(\Omega_A).$$

(ii) For any $p \in (0, 1)$ there exists a unique invariant measure for the infinite volume Markov process with $\Lambda = \mathbf{Z}$.

Proof.

(i) We use the coupling (a) and (c) with $\Lambda' = A$. It follows that $\sigma_t^\Lambda(x) = \sigma_t^A(x)\ \forall x \in A$. Thus the result follows trivially.

(ii) Because of (i) on any cylindrical event any infinite volume invariant measure coincides with the finite volume invariant measure and therefore the result follows. □

We conclude this section with a final result valid in the one-dimensional case which will be most useful in the next section when we will prove the basic probabilistic estimate. Given a subinterval A of Λ and a time interval $[0,t]$ we denote by $\Omega_A^{[0,t]}$ an arbitrary event that depends on the process $\sigma_{s'}(x)$ only for $s' \in [0,t]$ and $x \in A$. In general s' may be any integer or half integer between 0 and t.

Theorem 2.2. Let A, B, C be disjoint subintervals of Λ such that B separates A from C. Then for any initial configuartion σ:

$$P(\Omega_A^{[0,t]} \cap \Omega_C^{[0,t]} \cap \{\forall s \in [0,t]\ \exists x \in B; \sigma_s(x) = 0\}) \le P(\Omega_A^{[0,t]})P(\Omega_C^{[0,t]}).$$

Proof. Let us choose the following generalization of the coupling (a) and (b′) among the dynamics in Λ and the dynamics in $\Lambda' = A \cup C$: if a cluster $I \subset \Lambda$ intersects $A(C)$ but not $C(A)$ then it will be updated exactly as $I \cap A$ $(I \cap C)$; in all the other cases the updating is as in the basic dynamics in Λ. Clearly under this coupling the probabilities of interest for us are unchanged. Moreover if $\sigma_s(x)$ is such that $\forall s \in [0,t] \exists x \in B$ such that $\sigma_s(x) = 0$ then it is quite clear that:

$$\sigma_s(x) = \sigma_s^A(x) \qquad \forall x \in A$$

and the same for C. Thus:

$$\begin{aligned}
&P(\Omega_A^{[0,t]} \cap \Omega_C^{[0,t]} \cap \{\forall s \in [0,t] \exists x \in B; \sigma_s(x) = 0\}) \\
&\quad = P(\{(\sigma_s^A)_{s\in[0,t]} \in \Omega_A^{[0,t]}\} \cap \{(\sigma_s^C)_{s\in[0,t]} \in \Omega_C^{[0,t]}\} \\
&\qquad \cap \{\forall s \in [0,t] \exists x \in B; \sigma_s(x) = 0\}) \\
&\quad \le P(\{(\sigma_s^A)_{s'\in[0,t]} \in \Omega_A^{[0,t]}\} \cap \{(\sigma_s^C)_{s\in[0,t]} \in \Omega_C^{[0,t]}\}) \\
&\quad = P(\{(\sigma_s^A)_{s\in[0,t]} \in \Omega_A^{[0,t]}\})P(\{(\sigma_s^C)_{s\in[0,t]} \in \Omega_C^{[0,t]}\}) \\
&\quad = P(\Omega_A^{[0,t]})P(\Omega_C^{[0,t]})
\end{aligned}$$

where we have used theorem 2.1.

3. A basic probabilistic estimate

In this section we will establish a basic probabilistic estimate that will allow us to derive in the following sections a series of interesting results on the unique invariant measure of the process and on the rate of approach to equilibrium. The result is stated in the next theorem:

Theorem 3.1. For any $p \in (0,1)$ there exist positive constants $m(p), C(p)$ such that for any integer n:

$$\sup_{\sigma} P(\sigma_t(x) = 1 \ \forall x \in [1,n]) \le C(p)\exp(-m(p)n) + n2^{-t}.$$

Proof. For brevity let $\Omega_{(n,t)}$ be the event considered in the theorem and let us preliminarily note that the event involved in the theorem depends only on the process for $x \in [1,n]$. Then, using theorem 1.2, we may restrict the dynamics to the interval $[1,n]$ avoiding possible problems caused by the infinite volume.

We now turn to the proof of the theorem. Whenever otherwise specified all the estimates on the probabilities of various events of interest for us will always be uniform in the initial configuration. As a warm up let us first discuss the easier case $p \ll 1$. The general case will be proved by a very similar argument after collecting variables into blocks of suitable size.

Let for any $x \in [1,n]$

$$h(x) = \sup(s \in [0,t], s \in \mathbb{N}; \sigma_{t-s'}(x) = \sigma_{t-s'-1/2}(x) = 1 \ \forall s' \le s-1). \tag{3.1}$$

Clearly if the process σ_s is such that, at time t, $\sigma_t(x) = 1 \ \forall x \in [1,n]$ then the variables $h(x)$ are well defined. A generic realization of the random variables $(h(x))_{x=1}^n$ will be denoted by γ.

Clearly γ can be looked upon as an histogram (see figure 1).

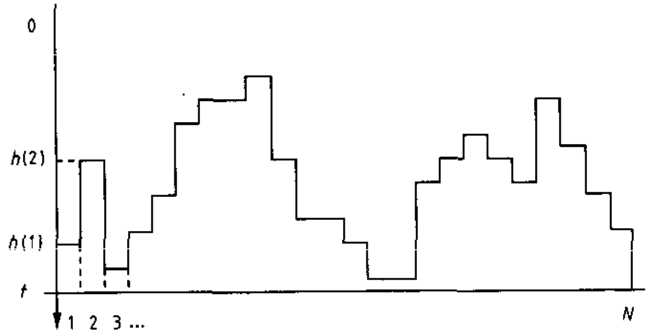

Figure 1.

We will denote by Γ_t the class of graphs γ such that their maximum height is less than t.

We will now estimate $P(\Omega_{n,t})$ by

$$P(\Omega_{n,t}) \le \sum_{\gamma \in \Gamma_t} P(\Omega_{n,t} \cap \{(h(x))_{x=1}^n = \gamma\}) + P(\exists x \text{ with } h(x) = t). \tag{3.2}$$

The second term on the right-hand side (RHS) of equation (3.2) is easily estimated by

$$nP(\sigma_s(0) = \sigma_{s-\frac{1}{2}}(0) = 1 \ \forall s \le t) \le n(\tfrac{1}{2})^t. \tag{3.3}$$

We now estimate the first term of (3.2). For a given curve γ let L be its total length, let V be its vertical length so that $L = V + n$, let k be the number of right angles in the curve and $P_1, \ldots, P_k$ be their positions measured as we move along the curve. Note that since any right angle has one horizontal side then k cannot exceed $(2n \wedge L)$. We claim that for any $\gamma \in \Gamma_t$ we have:

$$P(\Omega_{n,t} \cap \{(h(x))_{x=1}^n = \gamma\}) \leq p^n (\tfrac{1}{2})^{V/2}. \tag{3.4}$$

Proof. By construction, for any x, we must have created at x at time $t - h(x) > 0$ a particle that has lasted up to time t. Since the process of creating the particles is uncorrelated in spacetime we obtain the factor p^n. Now let for any half integer time s, smaller than $\sup_x(h(x)), n(s)$ be the number of intersections between γ and the line $s =$ constant. It is easy to check that $\sum_s n(s) = V$. By construction, if $\{h(x)\} = \gamma$ then for any such s we have $n(s)/2$ clusters that do not die and this last event has probability

$$(\tfrac{1}{2})^{\sum n(s)/2} = (\tfrac{1}{2})^{V/2} \tag{3.5}$$

and (3.4) follows.

We can now estimate $\sum_{\gamma \in \Gamma_t} P(\Omega_{n,t} \cap \{(h(x))_{x=1}^n = \gamma\})$ by

$$\sum_{L \geq n} \sum_{k=1}^{L} \binom{L}{k} 2^{k/2} p^n (\tfrac{1}{2})^{V/2}. \tag{3.6}$$

The factor $2^{k/2}$ in (3.6) comes from the fact that at each right angle after a horizontal step the graph can either increase or decrease. Since $k \leq 2n$ we estimate the RHS of (3.6) by

$$p^{n/2} \sum_{L \geq n} \sum_{k=1}^{L} \binom{L}{k} 2^{k/2} p^{k/4} (\tfrac{1}{2})^{(L-n)/2} = p^{n/2} \sum_{L \geq n} (\tfrac{1}{2})^{(L-n)/2} (1 + (\sqrt{2} p^{1/4}))^L \tag{3.7}$$

which for p small enough is bounded above by

$$p^{n/2} (1 + (\sqrt{2} p^{1/4}))^n C(p) \tag{3.8}$$

for a suitable constant $C(p)$ independent of n. The assertion of the theorem in this case follows with $m(p) = -\log(p^{1/2}(1 + (\sqrt{2} p^{1/4}))$.

We now turn to the general case with p arbitrary. When p is not sufficiently small the expansion previously given no longer works. To extend the result to p close to 1 the objective is to replace single sites by blocks of length l and to perform the previous expansion over the blocks. The key ingredient is that, upon renormalization, we will be able to associate, for a sufficiently large number of blocks, a weight $p(l)$ which goes to zero as the length l of the blocks diverges. It will turn out that in order to obtain a convergent expansion we will need to take $l = k/(1-p)$ with k a large constant independent of p.

Definition 1. Let for $l \in \mathbf{N}$ $n_l = [n/l]$ and let $B_i = [il, (i+1)l] \cap \mathbf{Z}$; we will denote by $\Omega_{n_l,t}$ the event $\{\sigma_t(x) = 1 \ \forall x \in B_i \ i = 0, \ldots, n_l\}$. Note that $\Omega_{n,t} \subset \Omega_{n_l,t}$. In analogy with (3.1) we will associate with each block B_i a height $h(i)$ defined by:

$$h(i) = \sup(s \in [0,t], s \in \mathbf{N}; \sigma_{t-s'}(x) = \sigma_{t-s'-1/2}(x) = 1 \ \forall x \in B_i, \forall s' \leq s-1). \tag{3.9}$$

A generic configuration of the random variables $\{h(i)\}_{i=1,\ldots,n_l}$ will be denoted by γ.

Definition 2. (a) i is a relative maximum for γ iff $h(i-1) < h(i)$ and $h(i) > h(d(i))$ where $d(i) = \min\{j > i; h(j) \neq h(i)\}$.

(b) i is a terrace for γ iff $h(i-1) = h(i) = h(i+1)$.

(c) i is a local minimum for γ iff $h(i-1) > h(i)$, $h(i) < h(d(i))$.

Definition 3. We denote by $\Gamma = \Gamma(i_1 \ldots i_k; j_0 \ldots j_k; p_1 \ldots p_q; \Delta_1 \ldots \Delta_k; \Delta'_1 \ldots \Delta'_k; h_1 \ldots h_q)$ the set of curves γ such that:

(1) γ has exactly k relative maxima at $i_1 \ldots i_k$;

(2) γ has exactly $k+1$ local minima at $j_0 \ldots j_k$;

(3) γ has exactly q terraces at $p_1 \ldots p_q$ with corresponding heights $h_1 \ldots h_q$;

(4) $h(i_m) - h(j_{m+1}) = \Delta_{i_m}$;

(5) $h(i_m) - h(j_m) = \Delta'_{i_m}$;

(6) $\sup_i h(i) \leq t - t_0(l)$ where $t_0(l) = (4|\log(1-p)|)^{-1} \log(l)$ (see figure 2).

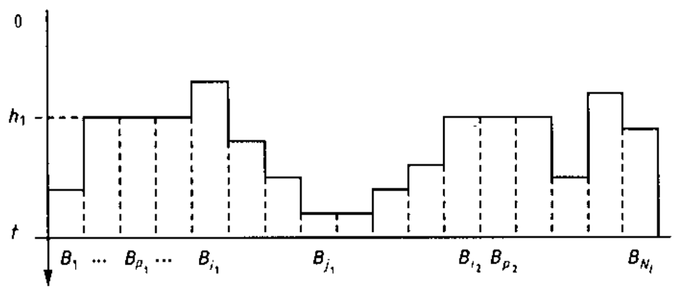

Figure 2.

We will now estimate $P(\Omega_{n,t})$ by

$$P(\Omega_{n,t}) \leq P(\Omega_{n_l,t}) \leq \sum_k \sum_n \sum_{(i_0 \ldots i_k; j_0 \ldots j_k)} \sum_{(p_1 \ldots p_q)} \sum_{(\Delta_1 \ldots \Delta_k; \Delta'_1 \ldots \Delta'_k)} \sum_{h_1 \ldots h_q} \times P(\Omega_{n_l,t} \cap \{\gamma \in \Gamma\}) + P(\sup_i h(i) > t - t_0(l)). \tag{3.10}$$

The second term on the RHS of (3.10) is estimated as in (3.3) by

$$n(l)(\tfrac{1}{2})^{t - t_0(l)}.$$

The following result, whose proof is for simplicity postponed to the appendix, will allow us to evaluate a limit for the first term on the RHS of (3.10).

Lemma 3.1. For any $p \in (0,1)$ there exists $l(p) > 0$ such that for any $l \in \mathbb{N}$, $l > l(p)$ there exists a positive constant $m(l,p)$ such that:

(i) $P(\Omega_{n_l,t} \cap \{\gamma \in \Gamma\}) \leq (\frac{1}{2})^{\sum_i (\Delta_i + \Delta'_i)/2}$

(ii) $P(\Omega_{n_l,t} \cap \{\gamma \in \Gamma\}) \leq \exp(-m(l,p)(k+q))$

(iii) $P(\Omega_{n_l,t} \cap \{\gamma \in \Gamma\}) \leq \exp\left(-\dfrac{m(l,p)(k+q)}{2}\right) (\frac{1}{2})^{\sum_i (\Delta_i + \Delta'_i)/4}$.

Furthermore $m(l,p) \to \infty$ as $l \to \infty$.

For simplicity we define $m' = m(l,p)/2$ and $m_0 = -\log(2)/4$. By using lemma 3.1 let us estimate the RHS of (3.10). To do that we first make the following elementary observations:

Remark 1. If we denote by n_i, n_i', $i = 1 \dots k$, the number of terraces of γ between j_{i-1}, i_i and i_i, j_{i+1} respectively, then the sum over the heights $h(i)$, $i = 1 \dots q$, in (3.10) can be estimated by

$$\Pi_i \left(\frac{(\Delta_i)^{n_i}}{n_i!} \right) \left(\frac{(\Delta_i')^{n_i'}}{n_i'!} \right).$$

This is so because for the n_i terraces in $[j_i, i_i]$ the heights are constrained to be non-decreasing and to lie between $h(j_i)$ and $h(i_i)$.

Remark 2. $\sum_i(\Delta_i + \Delta_i') \geq n_l - q/2$. To prove this simple inequality one first observes that this sum represents the total vertical length of the curve γ; moreover any block B_i which is not a terrace has at least one neighbour with a different height. Thus the lateral length has to be bigger than the half of the total number of blocks minus the terraces.

With these observations in mind and using lemma 3.1 we obtain:

$$\begin{aligned} \text{RHS of (3.10)} \leq \sum_k \sum_q \exp\left(-m'(k+q)\right)\tbinom{n_l}{k}^2 \tbinom{n_l}{q} \exp\left(-\frac{m_0(n_l - q)}{2}\right) \\ \times \Pi_i \left(\sum_{\Delta_i} \frac{(\Delta_i)^{n_i}}{n_i!} \exp\left(-\frac{m_0(\Delta_i)}{2}\right) \right) \left(\sum_{\Delta_i'} \frac{(\Delta_i')^{n_i'}}{n_i'!} \exp\left(-\frac{m_0(\Delta_i')}{2}\right)\right). \end{aligned} \tag{3.11}$$

The RHS of (3.11) is now easily estimated if we note that

$$\sum_{\Delta > 0} \frac{(\Delta)^q}{q!} \exp\left(-\frac{m_0 \Delta}{2}\right) \leq c(m_0)^q$$

for a suitable constant $c(m_0)$. The final result is:

$$\begin{aligned} \exp\left(-\frac{m_0 n_l}{2}\right) \left(1 + c(m_0) \exp\left(-m' + \frac{m_0}{2}\right)\right)^{n_l} \left(1 + \exp\left(\frac{-m'}{2}\right)\right)^{2n_l} \\ \leq \exp\left(-\frac{m_0}{4} n_l\right) \end{aligned} \tag{3.12}$$

if $m'(l,p)$ is large enough.

Clearly this estimate proves the theorem since $n_l \geq n/l - 1$.

4. Convergence to the invariant measure

We briefly discuss in this section the convergence of the distribution of the process at time t to the unique invariant measure of the process as t tends to infinity.

To be precise let $f : \{0,1\}^{\mathbf{Z}} \to \mathbf{R}$ be a cylindrical observable and let $S_f \subset \mathbf{Z}$ be the smallest subset of $\mathbf{Z}$ such that f does not depend on the values of the configuration σ outside S_f. There is no loss of generality in assuming that $S_f = [0, L]$. Now let Λ be such that $[0, L] \subset \Lambda$. We want to study the time behaviour of the quantity:

$$\rho_f(t) = \sup_\sigma |E_\sigma f(\sigma_t^\Lambda) - \mu_\Lambda(f)| \tag{4.1}$$

where as before $E_\sigma f(\sigma_t^\Lambda)$ and $\mu_\Lambda(f)$ denote the expectation value of f over the basic dynamics in Λ and with respect to the invariant measure in Λ respectively. The decay property of this quantity as $t \to \infty$ may provide a measure of how fast the process reaches its invariant measure.

As explained in section 1, this quantity does not change if one replaces Λ with the interval $[0, L]$; therefore from now on we will take $\Lambda = [0, L]$.

As we have already noticed in the previous section, by standard arguments for Markov chains with a finite number of states it follows that $\rho_f(t)$ decays exponentially to zero as t tends to infinity. More precisely there exist two positive constants $C_{f,L}$ and m_L such that

$$\rho_f(t) \leq C_{f,L} \exp(-m_L t). \tag{4.2}$$

We want to show that for any p the decay constant m_L can be chosen to be independent of L. More precisely we will prove:

Theorem 4.1. For any $p \in (0, 1)$ there exist two positive constants $m(p)$ and $\alpha(p)$ such that for any $t \geq \alpha(p)L$ we have

$$\rho_f(t) \leq \sup_\sigma |f(\sigma)| \exp(-m(p)t).$$

Proof. Using the invariance of the measure μ_Λ we estimate $\rho_f(t)$ by

$$\rho_f(t) \leq \sup_{\sigma,\eta} |E_\sigma f(\sigma_t^\Lambda) - E_\eta f(\eta_t^\Lambda)|. \tag{4.3}$$

We now couple the two dynamics starting from σ and η in such a way that with large probability after a time proportional to L they will become identical. The coupling goes as follows.

(i) At each time $s \leq t$ we use exactly the same random variables $\nu(x, s)$ $\xi(s, I)$ for both evolutions.

(ii) Let $x(s)$ be such that $\sigma_s(x) = \eta_s(x)$ $\forall x \leq x(s)$ and $\sigma_s(x(s) + 1) \neq \eta_s(x(s) + 1)$ provided that $x(s) \leq L - 1$. $x(s)$ may not exist. If such an $x(s)$ exists then for any cluster I containing $x(s)$ but not entirely contained in $[0, x(s)]$ we replace the variable $\xi(s, I)$ with the variable $\xi(s, I \cap [0, x(s)])$.

It is easy to check that these rules are indeed a coupling and that the sites $x(s)$ are non-decreasing in s, i.e. $x(s + 1) \geq x(s)$. It is also clear that if at time s $x(s)$ does not exist and at time $s + 1$ a particle is created at the origin then necessarily $x(s + 1) \geq 1$.

Moreover given $x(s)$ the probability that $x(s+1)$ is at least $x(s)+1$ is greater than p since the same particles are created at the same time for both the evolutions starting from σ and η. Therefore the site $x(s)$ moves to the right with an average velocity at least p. Standard large deviation estimates together with the strong Markov property now easily give that

$$\sup_{\sigma,\eta} P(x(t) < L) \leq \exp(-m(p)t) \tag{4.4}$$

for a suitable constant $m(p) > 0$ provided that t is, for example, greater than $2L/p$.

Since the RHS of (4.3) is estimated by

$$\sup_{\sigma} |f(\sigma)| \sup_{\sigma,\eta} P(x(t) < L) \tag{4.5}$$

the theorem follows. □

Remark. It follows from this result that the typical time scale needed to reach equilibrium in an interval of size L is at most proportional to L. For p small, the multiscale technique developed in [7] to study the same problem for the SW dynamics for the Ising model provides a much better result namely a power of $\log(L)$. We actually believe that this result should hold for all p but we have not been able to prove it.

5. Decay of correlations for the invariant measure in the one-dimensional case

In this section we use the basic probabilistic estimate proved in section 2 to show that connected correlations of the unique invariant measure decay exponentially in the one dimensional case.

Let A and B be finite intervals of lengths $|A|$ and $|B|$ separated by the interval D of length L. Let also f and g be local observables depending only on the value of the configuration $\sigma(x)$ for x in A and in B respectively and let $|f|_\infty$ denote the sup norm. We will write $\langle f; g\rangle$ for the expression:

$$\int d\mu(\sigma) f(\sigma) g(\sigma) - \left(\int d\mu(\sigma) f(\sigma)\right)\left(\int d\mu(\sigma) g(\sigma)\right). \tag{5.1}$$

Then we have:

Theorem 5.1. For any $p \in [0,1]$ there exist positive constants $m'(p)$, $L(p, |A|, |B|, f, g)$ such that for any $L > L(p, |A|, |B|, f, g)$:

$$\langle f; g\rangle \leq \exp(-m'(p)L).$$

Remark. The constant $L(p, |A|, |B|, f, g)$ does not depend on the mutual distance between A and B

Proof. We first recall the result proved in the previous section: in any finite volume Λ the basic dynamics, uniformly in the initial configuration, after a time proportional to the length L of Λ, approaches the invariant measure corresponding to that volume exponentially fast in time with a rate λ independent of L. In other words:

$$\sup_{\sigma} |E_\sigma f(\sigma_t) - \int d\mu_\Lambda(\sigma) f(\sigma_t)| \leq |f|_\infty \exp(-\lambda t) \tag{5.2}$$

for any $t > c(p)L$ where $c(p)$ is a positive constant.

Now let $m(p)$ be the positive constant of theorem 3.1 and let $T = kL$, with $k > 0$ to be chosen later on. We write:

$$\langle f; g\rangle = \lim_{t\to\infty} E_\sigma[f(\sigma_t)g(\sigma_t)] - \left(\int d\mu(\sigma)f(\sigma)\right)\left(\int d\mu(\sigma)g(\sigma)\right). \tag{5.3}$$

The second term on the RHS of (5.3), because of theorem 1.1, is equal to

$$\left(\int d\mu_A(\sigma)f(\sigma)\right)\left(\int d\mu_B(\sigma)g(\sigma)\right) \tag{5.4}$$

where μ_A is the invariant measure for the basic dynamics restricted to A and the same for μ_B. In order to compute the first term let us denote by η_s^A or η_s^B the process at time s $\in [0, T]$ evolving with the basic dynamics in A or in B starting at $s = 0$ from the configuration σ_{t-T} reached by the basic dynamics in $\mathbf{Z}$ at time $t - T$. The same simple argument used in the proof of theorem 1.1 shows that

$$\begin{aligned}\sup_\sigma |E_\sigma[f(\sigma_t)g(\sigma_t)] - E_\sigma E^{A\cup B}_{\sigma_{t-T}}[f(\eta_T^A)g(\eta_T^A)]| \\ \leq\ 2|f|_\infty|g|_\infty \sup_\sigma P(\exists s \in [t-T, t]; \sigma_s(x) = 1\ \forall x \in D)\end{aligned} \tag{5.5}$$

where $E^{A\cup B}$ denotes the expectation over the basic dynamics in A and in B. Note that by construction $E^{A\cup B}()$ is a product measure $E^A()E^B()$.

In turn, using theorem 3.1, the RHS of (5.5) is estimated by

$$kL[C(p)\exp(-m(p)L) + 2^{-(t-T)}] \tag{5.6}$$

with $C(p)$ and $m(p)$ as in theorem 2.1.

Moreover, using (5.2),

$$\begin{aligned}\left| E^{A\cup B}_{\sigma_{t-T}}[f(\eta_T^A)g(\eta_T^A)] - \int d\mu_A(\sigma)f(\sigma)\int d\mu_B(\sigma)g(\sigma)\right| \\ \leq\ |f|_\infty|g|_\infty \exp(-\lambda kL) + |g|_\infty|f|_\infty \exp(-\lambda kL)\end{aligned} \tag{5.7}$$

uniformly in the configuration σ_{t-T}, provided that kL is greater than $c(p)|A|$ and $c(p)|B|$.

Putting (5.3)–(5.7) together we finally get

$$\begin{aligned}\langle f; g\rangle \leq kL[C(p)\exp(-m(p)L) + 2^{-(t-T)}] \\ + |f|_\infty|g|_\infty \exp(-\lambda kL) + |g|_\infty|f|_\infty \exp(-\lambda kL).\end{aligned} \tag{5.8}$$

It is clear that by taking the constant k large enough depending on λ we can find a constant $L = L(|A|, |B|, p, f, g)$ such that for all sufficiently large t and for all $L \geq L(|A|, |B, |, p, f, g)$ the RHS of (5.8) is less or equal than:

$$\exp(-m(p)L/2)$$

and the theorem follows. □

6. The multi-dimensional case

In this section we will analyse in detail the d-dimensional case, $d > 1$; for simplicity we will mainly discuss the case $d = 2$ but the same arguments apply to any $d \geq 2$. As we will see for $d > 1$ the model becomes much more involved than the one-dimensional case and new kind of behaviours for the quantities of interest for us will appear.

Let us first give some notation: we will denote by $|C|$ the cardinality of a cluster C and by ∂C the set $\{x \in C; \exists y \notin C \text{ with } |x - y| = 1\}$. As in the one-dimensional case, we will first estimate the probability of having a large cluster completely full of particles because of the important role played by this quantity. We denote by Q_n the square of side n on the lattice , $Q_n = [1, n]^2$, and we define the event:

$$\Omega_{n,t} = \{\sigma_t(x) = 1 \text{ for any } x \in Q_n\}. \tag{6.1}$$

As it will be clear later on, we now have to use completely different techniques from those used in the one-dimensional case. In particular, even for very small values of p, an estimate of $P(\Omega_{n,t})$ cannot be obtained by means of an expansion in skyscrapers like the one used in the one-dimensional case. If such an expansion could work than it would imply an upper bound for the probability in question exponentially small in the volume of Q_n i.e. in n^2. To our surprise, however, it is possible to prove (see section 6.1) a lower bound which is only exponentially small in n for all p! In section 6.2 we will greatly improve such a bound for p close to one, to obtain upper and lower bounds for $P(\Omega_{n,t})$ which decay only as a certain powers of $1/n$.

6.1. Bounds on $P(\Omega_{n,t})$ valid for all p

We begin by proving a lower bound:

Theorem 6.1. For all $p \in (0, 1)$ and $n > 4$ let $T(n,p) = 4n/p$; then for all $t > T(n,p)$ we have

$$P(\Omega_{n,t}) \geq p^{5n}(\tfrac{1}{2})^{kn/p} \tag{6.2}$$

for some positive constant k.

The idea of the proof is very simple: at time $t - T(n,p)$ we create $4n$ particles on the boundary of Q_n and then we impose the condition that these particles survive for a time $T(n,p)$. The probability of this event is clearly only exponentially small in n. We will then show that, for n sufficiently large, if the cluster formed by the particles at the boundary of Q_n survives for a time as long as $T(n,p)$, then it is able to propagate inside Q_n and to fill it up completely with a probability not smaller than a negative exponential in n.

Proof of theorem 6.1. We define the following event:

$$\Theta_{r,t} = \{\sigma_r(x) = 1\ \forall x \in \partial Q_n \text{ and } \forall s \in [r,t]\ \xi(C_\partial(s), s) = 1\} \tag{6.3}$$

where $C_\partial(s)$ is the cluster at time s containing ∂Q_n.

For each realization of the dynamics satisfying $\Theta_{r,t}$ we have that $C_\partial(s)$ is well defined for all $s \leq t$ and that the number $n(s)$ of particles in $C_\partial(s)$ is a non-decreasing function of time:

$$n(s+1) \geq n(s).$$

We now introduce events that force $n(s)$ to increase at a certain rate:

$$\Gamma_{r,t} = \{\Theta_{r,t} \text{ holds and } \forall s \in (r,t]\ n(s) \geq n(s-1) + \frac{p}{2}\sqrt{n^2 - n(s-1)}$$
$$\text{if } n(s-1) < n^2 - n\}.$$

The probability of the event $\Gamma_{r,t}$ is estimated by

Lemma 6.1.

$$P(\Gamma_{r,t}) \geq p^{4n}[\tfrac{1}{2}(1 - \mathrm{e}^{-c_p\sqrt{n}})]^{t-r}$$

where c_p is a constant proportional to p.

Proof of lemma 6.1. We will estimate $P(\Gamma_{r,t})$ recursively by means of the Markov property. We write

$$P(\Gamma_{r,t+1}) \geq E[\chi(\Gamma_{r,t}) \inf_{\sigma_t;\Gamma_{r,t} \text{ holds}} E_{\sigma_t}\chi(\xi(C_\partial, t+1) = 1)$$
$$\chi(n(t+1) \geq n(t) + \frac{p}{2}\sqrt{n^2 - n(t)} \text{ if } n(t) < n^2 - n)]. \tag{6.4}$$

If we now denote by B_t the set

$$\{x; x \notin C_\partial(t) \text{ and } \exists y \in C_\partial(t); |x-y| = 1\}$$

we have the following simple geometric estimate proven later (see lemma 6.2):

$$|B_t| = |\partial(Q_n \backslash C_\partial(t))| \geq \sqrt{n^2 - n(t)}. \tag{6.5}$$

Moreover a simple exponential Chebyshev inequality shows that

$$P\ (\text{in } B_t \text{ we create at least } \frac{p}{2}|B_t| \text{ particles at time } t)\ \geq 1 - \mathrm{e}^{-c_p|B_t|} \tag{6.6}$$

where c_p is a constant proportional to p ($c_p \geq p/12$)

Now using (6.5) and (6.6) we can estimate from below the RHS of (6.4) by

$$P(\Gamma_{r,t})(\tfrac{1}{2}(1 - \mathrm{e}^{-c_p\sqrt{n}})).$$

If we now iterate the above lower bound $t-r$ times and observe that

$$P(\Theta_{r,r}) \geq p^{4n}$$

we get the lemma. It remains to prove (6.5). This in turn follows immediately from the definition of B_t and the next lemma:

Lemma 6.2. Let us define

$$f(n) = \min_{A \subset \mathbf{Z}^2; |A|=n} |\partial A|. \tag{6.7}$$

Then

$$f(n) \geq 4\sqrt{n} - 7$$

and, for $n > 5$, this implies $f(n) > \sqrt{n}$.

Proof of lemma 6.2. Let l be such that $l^2 < n \leq (l+1)^2$; then

$$\begin{aligned} f(n) &\geq 4l - 3 \qquad \text{if } l^2 < n \leq l(l+1) \\ f(n) &\geq 4l - 1 \qquad \text{if } l(l+1) < n \leq (l+1)^2 \end{aligned}$$

that is

$$f(n) \geq 4(l+1) - 7 \geq 4\sqrt{n} - 7. \tag{6.8}$$

In order to complete the proof of the theorem we need first to discuss the consequences of the recursion inequality $n(s) \geq n(s-1) + p/2\sqrt{n^2 - n(s-1)}$ implied by the event $\Gamma_{r,t}$. It is trivial to verify the following:

Lemma 6.3. The inequality

$$n(t+1) \geq n(t) + A\sqrt{B - n(t)} \tag{6.9}$$

with $n(0) = C$, $A > 0, B > C \geq 0$, implies that $n(t)$ is increasing for $t \leq t^* = 2/A\sqrt{B-C} + \frac{1}{2}$ and $n(t^*) \geq B$.

If we apply the lemma with $A = p/2$, $B = n^2$ and $C = 4n$ we can conclude that $n(t)$ is increasing for $t < (4/p)\sqrt{n^2 - 4n} + \frac{1}{2}$ and it is equal to $n^2 - n$ for some

$$t < (4/p)\sqrt{n^2 - 4n} < T(n,p). \tag{6.10}$$

We can now complete the proof of the theorem. Using lemmas 6.1 and 6.3 and the strong Markov property we get

$$P(\Omega_{n,t}) \geq P(\Omega_{n,t} \cap \Gamma_{t-T(n,p)-1,t-1}) \geq p^{4n}[\tfrac{1}{2}(1 - e^{-c_p\sqrt{n}})]^{T(n,p)}p^n$$

which concludes the proof of the theorem.

We now turn to the proof of an upper bound. Let for all $p \in (0,1)$ $t(n) = \alpha \log n$ with $\alpha = 4|\log(1-p)|^{-1}$. Then we have:

Theorem 6.2.

$$P(\Omega_{n,t}) \leq 2n^{-k} \qquad \text{for any } t > t(n)$$

with $k = \alpha|\log \frac{1}{2}|$.

Proof of theorem 6.2. We define the following event:

$$\Lambda_s \equiv \{\exists \text{ a connected cluster } C \subset Q_n; \sigma_s(x) = 1, \forall x \in C \text{ and } |C| > n^2/2\} \tag{6.11}$$

next we write:

$$P(\Omega_{n,t}) = P(\Omega_{n,t} \cap \Lambda_{t-t(n)}) + P(\Omega_{n,t} \cap \{\Lambda_{t-t(n)} \text{ does not hold}\}). \tag{6.12}$$

The second term in the RHS of (6.12) can be estimated by $\exp(-n^{1/2})$. In fact the set

$$D_{t-t(n)} \equiv \{x \in Q_n; \sigma_{t-t(n)}(x) = 0\} \tag{6.13}$$

contains at least n points, otherwise $\Lambda_{t-t(n)}$ would be true, and it must be filled of particles within an interval of time of length $t(n)$ for $\Omega_{n,t}$ to hold. The probability of this last event is given by the following lemma:

Lemma 6.3. Let $s \leq \alpha \log n$ and let $D \subset \mathbf{Z}^2$ with $|D| \geq k'n$ *then*

$$\sup_{\sigma; \sigma_0(x)=0, \forall x \in D} P(\sigma_s(x) = 1 \ \forall x \in D) \leq \exp(-\kappa' n^{1/2}). \tag{6.14}$$

Proof of lemma 6.3.

$$\begin{aligned}
\sup_{\sigma; \sigma_0(x)=0, \forall x \in D} P(\sigma_s(x) = 1 \ \forall x \in D) &\leq P(\forall x \in D \exists s' \leq s; \nu(x, s') = 1) \\
&\leq (1-(1-p)^s)^{|D|} \leq \exp - \{|D|(1-p)^s\} \leq \exp - \{\kappa' n e^{s(\log(1-p))}\} \\
&= \exp - \{\kappa' n n^{-2\alpha|\log(1-p)|}\} = \exp(-\kappa' n^{1/2}).
\end{aligned}$$

We now turn to the estimate of the first term of the RHS of (6.12). We write

$$\begin{aligned}
P(\Omega_{n,t} \cap \Lambda_{t-t(n)}) &= P(\Omega_{n,t} \cap \Lambda_{t-t(n)} \cap_{s=t-T(n)}^{t} \Lambda_s) \\
&\quad + P(\Omega_{n,t} \cap \Lambda_{t-t(n)} \cap \{\exists s \in (t - T(n), t]; \Lambda_s \text{ does not hold}\}).
\end{aligned} \tag{6.15}$$

If we observe that in the square Q_n there is at most one connected cluster C with $|C| \geq n^2/2$ completely filled with particles at any given time s, then the event described in the first term in the RHS of (6.15) implies that the cluster of size greater than $n^2/2$ which was present at time $t - T(n)$ did not die out for a time interval of length $T(n)$. The probability of this last event is bounded by $(\frac{1}{2})^{T(n)}$. The event described in the second term of the RHS of (6.15) instead implies that at some time s between $t - T(n)$ and t a cluster in Q_n of size greater than $n^2/2$ died out but that nevertheless it has been filled with particles within time t. The probability of this last event is again estimated via lemma 6.3 to get the bound $\exp(-k'n^2)$. The theorem now follows from the definition of $T(n)$. $\square$

6.2. Bounds on $P(\Omega_{n,t})$ valid for $p \sim 1$

For p sufficiently large the estimate given in theorems 6.1 can be improved. We have

Theorem 6.3. There exists a constant k sufficiently large such that for any $p > p_0$ and for any $t > 2T(n)$ where $T(n) = 4/p\alpha \log n$ we have

$$P(\Omega_{n,t}) \geq n^{-k\alpha(p)}$$

where $\alpha \geq 4/|\log(1-p)|$ and p_0 is the critical probability for site-percolation in two dimensions.

Sketch of the proof of the theorem 6.3. By the definition of our dynamics at each time t

$$\{x; \sigma_t(x) = 1\} \supseteq \{x; \nu(t,x) = 1\} \tag{6.16}$$

and therefore we can estimate our configuration with a configuration of the site-percolation model. As it is well known (see e.g. [14]) for this last model there exists a critical probability p_0 such that for $p > p_0$ with probability one there exists a unique infinite connected cluster of particles with 'holes' which are large only with small probability.

This enables us to conclude that at each time, with probability one, there exists an infinite cluster of particles C_∞ and with probability larger than $1 - n^{-\beta(\alpha)}$ the sites contained in the square Q_n which are not in C_∞ is a union of disjoint sets (holes) each one of size smaller than $\alpha \log n$, where $\alpha > \alpha(p)$ with $\alpha(p) \to 0$ as $p \to 1$ and $\beta \to \infty$ if $\alpha \to \infty$.

The strategy now follows that used in the previous section: we impose that the cluster C_∞ which is present at the initial time $t - 2T(n)$ survives for a time $2T(n)$ so that each hole present at time $t - 2T(n)$ shrinks during the time interval $[t - 2T(n), t - T(n)]$ by the same mechanism explained in theorem 6.1 in such a way that at time $t - T(n)$ each cluster of empty sites inside Q_n consists of at most constant $\times \log(n)$ sites. These remaining empty sites cannot be filled with particles in one step without paying too much in the probability but they can be eliminated in the last time interval $[t - T(n), t]$ by a simple procedure with probability close to one because of our choice of $T(n)$.

The implementation of these ideas is straighforward but quite lengthy and technical and we therefore decided to skip it.

7. Ergodicity in the case of large p

In this section we study the problem of convergence to equilibrium and uniqueness of the invariant measure in two dimensions for p close to one. We will work directly on the infinite lattice. For large values of p, at each time t, an infinite cluster of particles will be present since this already occurs for the independent process that creates the particles. Using standard results from percolation (see e.g. [10]) it follows that the infinite cluster will always be unique with probability one for each time t.

Our main result is the following:

Theorem 7.1. There exists $p_0 \in (0,1)$ such that for any $p \in (p_0, 1]$ there exists a unique invariant measure μ for the process; moreover there exists a positive constant γ such that for any cylindrical observable f we have

$$\sup_{\sigma} \left| \int \mathrm{d}\mu(\eta) f(\eta) - E_\sigma f(\sigma_t) \right| \le C_f \exp(-\gamma t) \tag{7.1}$$

where C_f is a suitable constant depending on f.

Proof. As is well known in the theory of stochastic particles systems the result follows if we can prove that for a suitable coupling there exists $p_0 \in (0,1)$ such that for any $p \in (p_0, 1]$ there exists a positive constant γ such that

$$\sup_{x} \sup_{\sigma,\eta} P(\sigma_t(x) \neq \eta_t(x)) \le e^{-\gamma t}. \tag{7.2}$$

We will therefore restrict ourselves to the proof of (7.2). The coupling that we will consider is the usual one, namely the *same* realizations of the random variables $\{\nu(x,t)\}$ and $\{\xi(C,t)\}$ are used to construct the basic dynamics for any initial condition, with only one extra rule:

If C_1 and C_2 are two different infinite clusters of $\mathbf{Z}^2$ then $\xi(C_1,t) = \xi(C_2,t)$.

This rule is compatible with our definition of the basic dynamics since at each time t and for any given initial configuration σ the infinite cluster in σ_t is unique.

Because of the definition of our coupling the event $\sigma_t(x) \neq \eta_t(x)$ implies that one of the following two events occurred:

(a) $\sigma_{t-1}(x) \neq \eta_{t-1}(x)$; or

(b) $\sigma_{t-1}(x) = \eta_{t-1}(x) = 1$ and in this case, if we denote by $C(\sigma,x)$ ($C(\eta,x)$) the clusters of particles containing x at time $t-1$ for the configuration σ and η, we have $C(\sigma,x) \neq C(\eta,x)$ and at least one of the two clusters say $C(\sigma,x)$ is finite. In both cases we can define a point $y_{t-1}(x)$ at which *the configurations are different at time* $t-1$, in the following way:

(i) $y_{t-1} = x$ in the first case;

(ii) $y_{t-1}(x)$ is such that $\sigma_{t-1}(y_{t-1}) \neq \eta_{t-1}(y_{t-1})$ and there exists a point $z \in C(\sigma,x)$ such that $|z - y_{t-1}| = 1$. Such a point exists since $C(\sigma,x)$ is finite and $C(\sigma,x) \neq C(\eta,x)$. This implies that if we denote with $\rho(t) = \sup_x \sup_{\sigma,\eta} P(\sigma_t(x) \neq \eta_t(x))$ then we have the following inequality:

$$\begin{aligned} \rho(t) &\le \sum_{R\ge 0} \sup_x \sup_{\sigma,\eta} P\ (\exists y \text{ such that } |x-y| = R \text{ and } y_{t-1}(x) = y \cap \sigma_t(x) \neq \eta_t(x)) \\ &\le\ ((1-p) + \sum_{L>4} P(A_L))\rho(t-1)) \end{aligned} \tag{7.3}$$

where A_L is the event:

$$A_L = \{\exists \text{ a closed loop of nearest neighbour sites around } x, \Gamma, \text{ with } |\Gamma| > L \\ \text{such that } \nu(t,z) = 0\ \forall z \in \Gamma\}.$$

Standard results of percolation theory give the estimate:

$$P(A_L) < \mathrm{e}^{-k(p)L}$$

with $k(p) \to \infty$ as $p \to 1$.

Thus for p sufficiently close to one we obtain the recursive inequality:

$$\rho(t) \le \beta\rho(t-1) \tag{7.4}$$

with $\beta < 1$. It is clear that (7.4) proves (7.2) and thus the theorem. □

Appendix. Proof of lemma 3.1

A1. Proof of (i)

Let us order in increasing order the heights $h(j)$ of the curve γ defined in (3.9), as follows (see figure A1):

(a) $\hat{h}_1 = \min_j(h(j))$

(b) $\hat{h}_i = \min_j(h(j); h(j) > \hat{h}_{i-1})$

(c) $C_i = \{$the set of maximal intervals J such that $\forall j \in J\ h(j) \geq \hat{h}_i\}$.

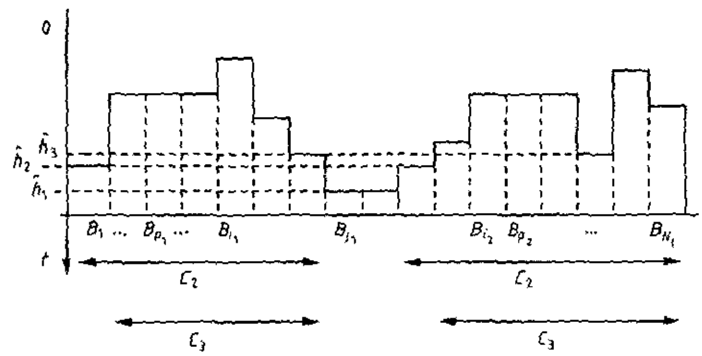

Figure A1.

We remark that by construction $C_i \subset C_{i-1}$ and that the elements of the class C_i resist up to time t and they are separated one from the other by an interval formed by an integer number of blocks of length l which is never totally occupied during the time interval $[t_i, t_{i-1}]$, where $t_i = t - h_i$ and by convention $t_0 = t$. Let us define P_j as:

$$P_j = P(\cap_{i \geq j}\{\text{each element } I \in C_i \text{ is never killed between } t_i, t_{i-1} \text{ and any two elements in } C_i \text{ are separated by an interval that is never totally occupied during the time interval } [t_i, t_{i-1}]\}). \qquad (A.1)$$

Then clearly P_1 is an upper bound for $P(\Omega_{n_l,t} \cap \{\gamma \in \Gamma\})$. Now let c_i denote the cardinality of C_i and let Σ_i denote the set of configurations σ such that at time t_i they are identically 1 on each element I of C_i and have at least one empty site between any two elements of C_i. Then, using the Markov property, we can estimate P_j by

$$P_j \leq P_{j+1} \sup_{\sigma \in \Sigma_j} P_\sigma(\{\text{each element } I \in C_j \text{ never die between } t_j, t_{j-1} \text{ and any two elements in } C_j \text{ are separated by an interval that is never totally occupied during the time interval } [t_j, t_{j-1}]\})). \qquad (A.2)$$

Now using theorem 3.1 the last factor in the RHS of (A.2) is bounded by

$$\left(\tfrac{1}{2}\right)^{c_j(t_{j-1}-t_j)} \qquad (A.3)$$

and therefore by iteration:

$$P_1 \leq \left(\tfrac{1}{2}\right)^{\Sigma c_j(t_{j-1}-t_j)}. \qquad (A.4)$$

If one observes that $\Sigma c_j(t_{j-1} - t_j)$ is equal to the lateral length of γ and that this last one coincides with $\sum_i(\Delta_i + \Delta_i')/2$ (see remark 2) we conclude that (A.4) proves (i) of lemma 3.1.

A2. Proof of (iii)

It simply follows by taking the geometric mean of (i) and (ii).

A3. Proof of (ii)

We first establish some simple results that we will need in the course of the proof.

Definition. For any integer l we set:
(i) $t_0(l) = (4|\log(1-p)|)^{-1}\log(l)$.
(ii) $P_\sigma(l) = P(\sigma_{t_0(l)}(x) = 1\ \forall x \in [1,[l/3]])$ and $P(l) = \sup_\sigma P_\sigma(l)$.
(iii) $\Pi(l) = P(\forall x \in [1,l]\nu(x,s) = 1)$ for some $s \in [1, 2t_0(l)]$.
We have the following simple results:

$$\Pi(l) = (1-(1-p)^{2t_0(l)})^l \leq \exp(-l^{1/2}). \tag{A.5}$$

Lemma A.1.

$$\lim_{l\to\infty} P(l) = 0.$$

Proof. We take l very large and we consider separately two different cases:
(1) There exists $x_0 < l/3 - l^{1/2}$ such that $\sigma(x) = 1\ \forall x \in [x_0, x_0 + l^{1/2}]$.
(2) There exists no x_0 satisfying this requirement.
In the first case we estimate $p_\sigma(l)$ by

$$P_\sigma(l) \leq P(x_0 \text{ never dies before } t_0(l)) + P(\{\exists s \leq t_0(l); \text{the cluster containing } x_0 \text{ is killed at time } s\} \cap \{\sigma_{t_0(l)}(x) = 1\ \forall x \in [1,[l/3]]\}) \tag{A.6}$$

note that the cluster containing x_0 has length greater than $l^{1/2}$ before it is killed. Therefore we can estimate the RHS of (A.6) by

$$(\tfrac{1}{2})^{t_0(l)} + t_0(l)\Pi(l^{1/2})$$

and in this case the lemma follows. □

In the second case we have that at the initial time σ contains m empty sites, with $m \geq l^{1/2}$. Therefore

$$P_\sigma(l) \leq \Pi(m) \leq \Pi(l)$$

and the lemma follows. □

Now let $B_1, \ldots, B_n$ be blocks of length l such that $\text{dist}(B_i, B_j) \geq 2l\ i \neq j$ and let $t_1, \ldots, t_n$ be integer times such that $t_i \geq t_0(l)$ where $t_0(l)$ is as earlier and let B_i^+ and B_i^- be the right neighbour block and the left neighbour block of B_i. Let also Ω_i denotes the event:

$$\Omega_i = \{B_i \text{ is totally filled with particles at time } t_i \text{ and} B_i^+, B_i^- \text{ contain at least one empty site at time } t_i - \tfrac{1}{2}\}. \tag{A.7}$$

Then we have

Lemma A.2. For any $p \in (0,1)$ there exists $l(p) > 0$ such that for any $l > l(p)$:

$$\sup_\sigma P_\sigma(\cap_{i=1\dots n}\Omega_i) \le f(l)^n$$

with $\lim_{l\to\infty} f(l) = 0$

Proof. Without loss of generality we may assume that l is an integer multiple of 3 and that $t_n > t_i$ $i = 1\dots n-k-1$ and $t_n = t_{n-1} = \dots = t_{n-k}$. Now let $\tilde{t}_n = t_n - t_0(l)$ and let $I = \{i \le n-k-1; t_i \ge \tilde{t}_n\}$. Using the Markov property we can write

$$\sup_\sigma P_\sigma(\cap_{i=1\dots n}\Omega_i) \le \sup_\sigma E_\sigma \Pi_{i\le n-k-1, i\notin I}\chi(\Omega_i) E_{\sigma_{\tilde{t}_n}} \Pi_{j=n-k\dots n}\chi(\Omega_j)\Pi_{j\in I}\chi(\Omega_j) \tag{A.8}$$

We will prove that

$$\Omega_j \subset \Omega'_j \cup \Omega''_j \ \forall j = n-k,\dots,n \tag{A.9}$$

where Ω'_j is a local event depending only on the variables $\nu(x,s), \xi(I,s)$ for (x,s) and (I,s) strictly contained in $(B_j^+ \cup B_j^- \cup B_j) \times [\tilde{t}_n, t_n]$, and Ω''_j has a simple estimate for its probability. In order to simplify the notations we will now denote by $\frac{1}{3}B_j$ $\frac{2}{3}B_j$ $\frac{3}{3}B_j$ the first third, the middle third and final third of the block B_j respectively. Given an interval $[a,b]$ we will also define the dynamics in $[a,b]$ with 'full boundary conditions' as the basic dynamics in $[a,b]$ with the constraint that clusters containing either a or b or both never disappear. It is quite clear that in $[a,b]$ we can couple the two dynamics in such a way that the process evolving with the dynamics with 'full boundary conditions' will always contain at least the same particles as the usual process evolving with the basic dynamics.

In these notations the event Ω''_j, $,\Omega'_j$ are defined as follows

$$\begin{aligned}\Omega''_j = \{\tfrac{2}{3}B_j &\text{ is filled with particles at time } t_n\} \cap \{\{B_j^- \cup \tfrac{1}{3}B_j\} \text{ was never completely} \\ &\text{occupied with particles between } \tilde{t}_n \text{ and } t_n - 1\} \cap \{\{B_j^+ \cup \tfrac{3}{3}B_j\} \text{ was} \\ &\text{never completely occupied with particles} \\ &\text{between } \tilde{t}_n \text{ and } t_n - 1\}\end{aligned} \tag{A.10}$$

$$\begin{aligned}\Omega'_j = \{\exists \tilde{t}_n < s_1 < s_2 \le t_n;\ &\text{one of the blocks } B_j^-, B_j^+, \tfrac{1}{3}B_j, \tfrac{3}{3}B_j \\ &\text{is filled with particles at time } s_2 \text{ starting empty at time } s_1 \\ &\text{by the dynamics with full boundary conditions on the} \\ &\text{boundary of the block}\}.\end{aligned} \tag{A.11}$$

The strategy now is the following: we first prove the inclusion (A.9) and then by using theorem 3.1, which can be applied by the definition of the event Ω''_j, we will factorize the probabilities appearing in (A.8).

In order to prove (A.9) we show that

$$\Omega_j \cap (\Omega'_j)^c \subset \Omega''_j. \tag{A.12}$$

Thus let us suppose that the event Ω_j' did not happen; then there are two different possibilities:

(a) In B_j^- and in B_j^+ there exists always an empty site for any time $s \in [\tilde{t}_n, t_n - 1]$.

(b) There exists a time interval $[r_1, r_2] \subset [\tilde{t}_n, t_n - 1]$ such that at least one of the blocks B_j^-, B_j^+ has been completely full of particles for any time s (half integers included) in $[r_1, r_2]$.

In (a) (A.12) is obvious.

In (b) let for definiteness B_j^+ be the block that is full during the time interval $[r_1, r_2]$.

Remark 1. There cannot exist any other interval $[r_1', r_2'] \subset [\tilde{t}_n, t_n - 1]$ with say $r_1' > r_2$ where B_j^+ is full, since in that case, by choosing $s_1 = r_2$ and $s_2 = r_1'$, Ω_j' would have occurred for the basic dynamics and *a fortiori* for the dynamics with full boundary conditions.

Remark 2. During the time interval $[r_1, r_2]$ the block $\frac{2}{3}B_j^+$ cannot be totally full of particles. In that case in fact at time $r_2 + \frac{1}{2}$ $\frac{3}{3}B_j^+$ would be empty and, therefore, by choosing $s_1 = r_2$ and $s_2 = t_n$, Ω_j' would have occurred.

Conclusion. For any $s \in [\tilde{t}_n, t_n]$ if $\frac{2}{3}B_j^+$ is full then there exists an empty site in B_j^+ and analogously for $\frac{1}{3}B_j^+$ and B_j^-. Therefore (A.12) follows.

We are now in a position to complete the proof of lemma 2.3. Using (A.9) we first estimate the RHS of (A.8) as

$$\sup_\sigma E_\sigma \Pi_{i \le n-k-1, i \notin I} \chi(\Omega_i) E_{\sigma_{\tilde{t}_n}} \Pi_{j=n-k \ldots n} (\chi(\Omega_j') + \chi(\Omega_j''))(\Pi_{j \in I} \chi(\Omega_j)). \tag{A.13}$$

At this stage we use the structure of the events Ω_j' Ω_j'' together with theorem 3.1 to estimate the RHS of (A.13) by

$$\begin{aligned} &\sup_\sigma E_\sigma \Pi_{i \le n-k-1, i \notin I} \chi(\Omega_i) E_{\sigma_{\tilde{t}_n}} (\Pi_{j \in I} \chi(\Omega_j)[\sup_{\sigma_{\tilde{t}_n}, j} P_{\sigma_{\tilde{t}_n}} (\Omega_j' \cup \Omega_j'')]^k \\ &\quad = \sup_\sigma E_\sigma \Pi_{i \le n-k-1,} \chi(\Omega_i) f(l)^k \end{aligned} \tag{A.14}$$

where $f(l) = \sup_{\sigma_{\tilde{t}_n}, j} P_{\sigma_{\tilde{t}_n}} (\Omega_j' \cup \Omega_j'')$.

Thus by iteration

$$\sup_\sigma P_\sigma (\cap_{i=1 \ldots n} \Omega_i) \le f(l)^n.$$

Thus it remains to prove that $\lim_{l \to \infty} f(l) = 0$.

We estimate $f(l)$ by

$$f(l) \le \sup_{\sigma_{\tilde{t}_n}, j} P_{\sigma_{\tilde{t}_n}} (\Omega_j') + P_{\sigma_{\tilde{t}_n}} (\Omega_j''). \tag{A.15}$$

The first and the second term in the RHS of (A.15), using the definition of Ω_j', Ω_i'', can be bounded by

$$t_0(l)^2 4\Pi(l)^{1/3} \qquad \text{and} \qquad P(l)$$

respectively. Thus $f(l)$ tend to zero as $l \to \infty$ because of (A.5) and lemma 3.2. The lemma is proved. □

We can now finish the proof of lemma 3.1. Clearly the blocks B_i that are either terraces or relative maxima for the curve $\gamma \in \Gamma$ satisfy the hypotheses of lemma 3.3 provided the maximum height of γ does not exceed $t - T(l)$, with the only possible exception of being at relative distance at least $2l$. This last problem is easily overcome by separating the set of relative maxima and terraces into two classes such that any two elements in the same class satisfy these distance condition, and then, using the Schwartz inequality, by estimating the probability of the intersection of the two classes with the product of the square root of the probability for each class. The probability of each class is estimated using lemma A.2. The final result is

$$P(\Omega_{n,t} \cap \{\gamma \in \Gamma\}) \leq f(l)^{k+n/2}$$

provided l is large enough depending on p and $f(l)$ is as in lemma 2.3. Lemma 2.1 is proved with $m(l,p) = -\log(f(l))$.

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